**Epidemiology:** Epidemiology is the study and analysis of the distribution (who, when, and where), patterns and [determinants](https://en.wikipedia.org/wiki/Risk_factor) of health and disease conditions in defined [populations](https://en.wikipedia.org/wiki/Population).

**Endemic:** An endemic is the outbreak of a disease in local level. It occurs when a disease exists permanently in a particular region, environment, or population.

Example: (1). Malaria in Africa.

(2). Polio in Afghanistan, Nigeria and Pakistan.

(3). Chicken pox among young school children in USA.

**Epidemic:** An epidemic is the outbreak in national level. It occurs when a disease spreads rapidly within a region or a group at a particular time.

Example: (1). Annual influenza epidemics follow a winter seasonal pattern in the United States with typical activity peaking during late December to early February.

(2). In 2003, the [severe acute respiratory syndrome](https://www.webmd.com/webmd/consumer_assets/controlled_content/healthwise/special/severe_acute_respiratory_syndrome_sars-overview_special_uf6068.xml) (SARS) epidemic took the lives of nearly 800 people worldwide.

**Pandemic:** A pandemic is the outbreak of a disease in global level. It occurs when an epidemic becomes very widespread and affects a whole region, a continent, or the entire world.

Example: (1). Spanish [influenza](https://www.webmd.com/cold-and-flu/ss/slideshow-cold-or-flu) killed 40-50 million people in 1918.

(2). Asian [influenza](https://www.webmd.com/cold-and-flu/cold-or-flu-quiz) killed 2 million people in 1957.

(3). Hong Kong [influenza](https://www.webmd.com/cold-and-flu/video/avoid-colds-flu) killed 1 million people in 1968.

**Epidemic models:** Let there is a disease in a given population at time . Then the population can be divided into three classes:

(a). , the number of susceptibles who may be infected by the disease.

(b). , the number of infectives who has the disease and can transmit it.

(c). , the number of those removed from the population by recovery, immunization, death, hospitalization or by any other means.

The epidemic models are various in types. We can classify them at least into two groups:

**(1).** **Disease without removal:** In this case, it is assumed that person one can never be removed from this disease. Therefore the total population always remains either in susceptible class or in infected class.

Example: (i). SI model

(ii). SIS model.

**(2). Disease with removal**: In this case, it is assumed that person one can be removed from the disease through recovery or death. This removal may be temporary or permanent.

Example: SIR model.

**SI model:** This is a simple deterministic model without removal. It has only susceptible and infected classes. Since, the number of susceptibles decreases and the number of infected increases due to infection, so the rate of decrease of susceptible and the rate of increase of infected both are proportional to the product of susceptibles and infected persons. If  be the total population size,  be the number of susceptibles and  be the number of infected persons in the population, then the model is defined as,

 

 

whereis a constant parameter. It is called infection rate.

Let  be the initial number of susceptible and  be the initial number of infected in the population. So that,

 

Since there is no removal in the population so we have,

 

Using (4) in (1), we get











Integrating,





 

By using the initial condition (3) in (5), we have



Putting the value of  in (5), we get













 

This represents the number of susceptible population at any time .

Again, using (4) in from (2), we have











Integrating,





 

By using the initial condition (3) in (7), we have



Putting the value of  in (7), we get













 

This represents the number of infected population at any time .

At , from (6) & (8), we have

 

and  

This shows that all persons will be infected at .

By putting the values of and  from (6) and (8) in (1), we get



 

The equation (11) is the equation of a curve which gives a relation between and . This curve is known as epidemic curve. The above curve is a symmetrical unimodel curve with maximum at  where  is obtained from (11) by













.

Thus, maximum time .

Thus, the rate of appearance of new cases rises rapidly to maximum at a time , depending on , ,,  and then falls to zero.

**SIS model:** This is another simple deterministic model without removal. It has also susceptible and infected classes. In this model, a susceptible person becomes infected and an infected person recovers and becomes susceptive again. If  be the total population size,  be the number of susceptibles and  be the number of infected persons in the population, then the model is defined as,

 

 

where is the infection rate and is the removal rate of infective.

Let  be the initial number of susceptible and  be the initial number of infected in the population. So that,

 

Since there is no removal in the population so we have,

 

From (2) and (4), we have



 







Integrating,



 

By using the initial condition (3) in (6), we have



Putting this value of  in (6), we get



















 ; 

When , then the equation (5) reduces as





Integrating,  

Using (3) in (8), we get



Putting the value of  in (8), we get





 

 

This represents the number of infected population at any time .

From (1) and (2), we get

 

When , then from (11), we have

















Integrating, 

 

Using (3) in (12), we get





Putting the value of  in (12), we get

 

Again, When , then from (11), we have







Integrating,  

Using (3) in (14), we have





Putting the value of  in (14), we get

 

 

This represents the number of susceptible population at any time .

**SIR model:** This is a simple deterministic model with removal. It has susceptible, infected and removal classes. In this model, the susceptible becomes infected and the infected can be removed from the population by death, isolation, recovery, hospitalization, immunization or by any other means. If  be the total population size,  be the number of susceptibles,  be the number of infected persons in the population and be the number of removed persons, then the model is defined as,

 

 

 

whereis the infection rate, is the removal rate of infective and  is relative removal rate.

Let  be the initial number of susceptible and  be the initial number of infected in the population. Since, at the beginning of epidemic, the number of removal is zero so

,  

Since there is removal in the population so we have,

 

This model is also called the classic Kermack-Mckendrick model.

From (1) and (2), we have







Integrating,  

Using (4) in (5), we get





Putting the value of  in (6), we get







 







This means that the total population may be susceptible as .

Equation (3) can be written as,



 ; [using (5)]

; [using (7)] 

If  is large and  is small, then



Now from (8), we have

; [neglecting higher order terms]



; [using (5)]











where 



Integrating, 







where 







 

This represents the number removal at time .

Again, from (2), we have





Integrating, 

 

Using (4) in (10), we get



Putting the value of  in (10), we get

 

This shows that the infection will die out i.e.  as  if . On the other hand, the infection will spread throughout the population  as  if .

**Questions**

**Q-01:** Discuss the SI epidemic model and show that the persons are ultimately infected.

**Q-02:** Discuss the deterministic models without removal and show that the infection spread throughout the population.

**Q-03:** In the epidemic model ,  in a closed population without removal, show that the persons are ultimately infected and that the rate of appearance of new cases rises rapidly to a maximum at time  and then falls to zero.

**Q-04:** Derive the differential equation for SI epidemic model and solve it.

**Q-05:** Derive the differential equation for SIS epidemic model and solve it.

**Q-06:** Derive the differential equation for SIR epidemic model and solve it for removal.

**Q-07:** Discuss the simple SIR epidemic model. Find the condition on which the infection (disease) will ultimately die out or spread throughout the population.

**Q-08:** Describe the Kermack-Mckendric SIR epidemic model. Define the relative removal rate . Discuss .

**Problems**

**Problem-01:** If the contact rate  be , the number of initial susceptible population  be  and the number of initial infected population  be , then determine,

(1) the number of susceptibles after  weeks.

(2) the density of susceptible when the rate of appearance of new of cases is a maximum.

(3) the time (in weeks) at which the rate of appearance of new cases is a maximum.

(4) the maximum rate of appearance of new cases, and

(5) the epidemic curve.

**Solution:** Given that, ,  and 

.

(1). The number of susceptibles after  weeks is,







.

(2) When the rate of appearance of new cases is maximum, then the density of susceptibles is,



(3) The time at which the rate of appearance of new cases is a maximum is,







.

(4) The maximum rate of appearance of new cases is,









(5) The epidemic curve is obtained by plotting  against the relation,



Here ,  and 

The curve is as shown as follows:

Figure-01

**Problem-02:** If the relative removal rate  be  and the number of initial susceptible population  be , then find the number of susceptible population left by the time  infected individuals are removed from circulation,

**Solution:** The number of susceptible left when  is individuals are removed from circulation.



Here, , and 







.

**Problem-03:** If the relative removal rate  be , the number of initial susceptible population  be , the number of initial infected population  be and the removal rate  be , then find the number of removal population after  weeks.

**Solution:** The removal population is defined as,

 

where  

and  

Here ,, , ,.

From (2), we get



From (3), we get



From (1), we get





.

**Problem-04:** Show that the angle between the vaccinated and non-vaccinated trajectories at SI is given by



and show that the directions of the two trajectories coincide only at  and the angle has different signs on the left and right of the line .

**Solution:** We have the set of equations,

 

 

 

 

where .

From equation (1) and (2), we get

 

Suppose the tangent at  to the vaccinated trajectories makes an angle  with - axis then from (5), we get

 

Also the tangent at  to the non-vaccinated trajectories makes an angle  with - axis then from (5), we get



 

Let,  be the angle between both types of trajectories, then we get













. 

This is the required angle between vaccinated and non- vaccinated trajectories at SI.

If , then from (8), we get







Hence the directions of two trajectories are coincide at . If , then from (8),  has negative sign and if ,  has positive sign.

Hence the has different signs on the left and right of the line . **(Proved)**

**Problem-05:** Describe the classic-Kermack-Mckendric SIR epidemic model. Discuss the spread of the infection according to this model and how it develops with time.

**Solution:** If  be the total population size,  be the number of susceptibles,  be the number of infected persons in the population and be the number of removed persons, then the SIR epidemic model is defined as,

 

 

 

whereis the infection rate, is the removal rate of infective and  is relative removal rate.

Let  be the initial number of susceptible and  be the initial number of infected in the population. Since, at the beginning of epidemic, the number of removal is zero so

,  

Since there is removal in the population so we have,

 

This model is also called the classic Kermack-Mckendrick model.

From (2), we have





Integrating, 

 

Using (4) in (10), we get



Putting the value of  in (10), we get

 

This shows that the infection will die out i.e.  as  if . On the other hand, the infection will spread throughout the population  as  if .

Again from (1) and (2) we have





Integrating,



 

Using (4) in (8) we get





Putting this value in (8) we have



 

From (2),  will be maximum if





, since 

Putting  in (9) we get



, since 

If  and , then the phase trajectory start with . Also in this case  increases from  and hence an epidemic ensures. If , then  decreases from  and as such no epidemic occurs.